

ON THE SINGULAR NEUMANN PROBLEM IN LINEAR ELASTICITY *

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Abstract. The Neumann problem of linear elasticity is singular with a kernel formed by the rigid motions of the body. There are several tricks that are commonly used to obtain a non-singular linear system. However, they often cause reduced accuracy or lead to poor convergence of the iterative solvers. In this paper, four different well-posed formulations of the problem are studied through discretization by the finite element method, and preconditioning strategies based on operator preconditioning are discussed. For each problem we derive preconditioners that are independent of the discretization parameter. Preconditioners that are robust with respect to the first Lamé constant are constructed for the pure displacement formulations, while a preconditioner that is robust in both Lamé constants is constructed for the mixed formulation. It is shown that, for convergence in the first Sobolev norm, it is crucial to respect the orthogonality constraint derived from the continuous problem. Based on this observation a modification to the conjugate gradient method is proposed that achieves optimal error convergence of the computed solution.

Key words. linear elasticity; rigid motions; singular problems; preconditioning; conjugate gradient

1. Introduction. The presented paper discusses numerical techniques for solving the singular problem of linear elasticity. Let $\Omega \subset \mathbb{R}^3$ be the body subjected to volume forces $f : \Omega \rightarrow \mathbb{R}^3$ and surface forces $h : \partial\Omega \rightarrow \mathbb{R}^3$. The body's displacement $u : \Omega \rightarrow \mathbb{R}^3$ is then found as a solution to

$$\begin{aligned} -\nabla \cdot \sigma(u) &= f && \text{in } \Omega, \\ \sigma(u) &= 2\mu\epsilon(u) + \lambda(\nabla \cdot u)I && \text{in } \Omega, \\ \sigma(u) \cdot n &= h && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

with $\mu > 0$, $\lambda \geq 0$ the Lamé constants of the material, I the identity matrix, $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ the strain and n the outward-pointing surface normal, see [1]. The system is used extensively in structural analysis [2], and is relevant in numerous applications for, e.g., marine engineering [3], biomechanics of brain [4], spine [5] or the mechanics of planetary bodies [6].

Due to the absence of a Dirichlet boundary condition that can anchor the body (coordinate system) in space, the problem can be solved if and only if the net force and the net torque on Ω are zero, i.e., the forces f , h satisfy the compatibility conditions

$$\begin{aligned} \int_{\Omega} f \, dx + \int_{\partial\Omega} h \, ds &= 0, \\ \int_{\Omega} f \times x \, dx + \int_{\partial\Omega} h \times x \, ds &= 0. \end{aligned} \tag{1.2}$$

With such compatible data the now solvable (1.1) is singular as any rigid motion can be added to the solution. We note that the space of rigid motions $z : \Omega \rightarrow \mathbb{R}^3$ such that $\epsilon(z) = 0$, consists of translations and rigid rotations and for a body in $3d$ the space is six-dimensional.

The ambiguity of the solution of (1.1) can be removed by adding constraints by means of Lagrange multipliers which enforce that the solution is free of rigid motions. When discretized, this approach yields an invertible saddle point system. Alternatively, discretizing (1.1) directly leads to a symmetric, positive semi-definite matrix with a six dimensional kernel. If (1.2) holds, such a system can be solved by the conjugate gradient (CG) method [7]. Finally, a common approach (here termed *pinpointing*) in engineering literature, e.g. [3], is to remove the nullspace by prescribing the displacement in selected points of $\partial\Omega$.

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In this work we focus on analysis of the Lagrange multiplier method and the conjugate gradient method for the singular problem (1.1). Well-posedness of both the methods is discussed and efficient preconditioners are established based on operator preconditioning [8]. Further, connections between the two methods and the question of whether they yield identically converging numerical solutions are elucidated.

The manuscript is structured as follows. In §2 the necessary notation is introduced and shortcomings of pinpointing and CG are illustrated by numerical examples. Section 3 discusses Lagrange multiplier formulation and two preconditioners for the method. Section 4 deals with the preconditioned CG method and two preconditioners are proposed. Further, it is revealed that if the variational origin of the discretized problem is ignored, the method, in general, will not yield convergent solutions. A variational setting is introduced to modify the CG to yield a convergent method. Section §5 discusses well-posedness and preconditioning of an alternative formulation of (1.1). The proposed formulation leads to a symmetric, positive definite linear system. In §3-5 we assume that λ and μ are of comparable magnitude in order to put the focus on proper handling of the rigid motions. In §6 we consider the case where $\lambda \gg \mu$. The focus here is on a well-known and simple technique to remove the problems of locking, namely the mixed formulation of linear elasticity. This formulation yields robust approximation and preconditioning in λ when care is taken of proper handling of the rigid motions. Finally, conclusions are drawn in §7.

2. Preliminaries. Let V be the Sobolev space of vector (or scalar or tensor) valued functions, which, together with their weak derivatives of order one, are in space $L^2(\Omega)$. We denote (\cdot, \cdot) the L^2 inner product of functions in V while $\|\cdot\|$ is the corresponding norm. The standard inner product of V is $(u, v)_1 = (u, v) + (\nabla u, \nabla v)$, $u, v \in V$ and $\|\cdot\|_1$ shall be the induced norm. For any Hilbert space V its dual space is denoted as V' and we use capital or calligraphy letters to denote operators, e.g. $A : V \rightarrow V'$ or $\mathcal{A} : (V \times V) \rightarrow (V \times V)'$. Finally, $\langle \cdot, \cdot \rangle$ is the duality pairing between V' and V .

The space \mathbb{R}^n is considered with the l^2 inner product $\mathbf{x}^\top \mathbf{y} = x_i y_i$ (invoking the summation convention), $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and the norm $|\mathbf{x}| = \sqrt{\mathbf{x}^\top \mathbf{x}}$. For clarity of notation bold fonts are used to denote vectors and operators(matrices) in \mathbb{R}^n that represent functions and operators from finite dimensional finite element approximation space $V_h \subset V$ with respect to its nodal basis $\{\phi_i\}_{i=1}^n$. The representations are obtained by mappings $\pi_h : V_h \rightarrow \mathbb{R}^n$ (the nodal interpolant) and $\mu_h : V_h' \rightarrow \mathbb{R}^n$ such that for $v \in V_h$, $f \in V_h'$

$$v = (\pi_h v)_i \phi_i \quad \text{and} \quad (\mu_h f)_i = \langle f, \phi_i \rangle. \quad (2.1)$$

We refer to [8, ch 6.] for a detailed discussion of the properties of the mappings, e.g. invertibility, and note here that $M : V_h \rightarrow V_h'$ is represented by a matrix $\mathbf{M} = \mu_h M \pi_h^{-1}$. In particular, the mass matrix \mathbf{M} , $M_{ij} = (\phi_j, \phi_i)$ represents the Riesz map with respect to the L^2 -inner product, $\langle Mu, v \rangle = (u, v)$, $u \in V_h$. On the other hand the duality pairing between V_h' and V_h is represented by the l^2 inner product $\langle f, v \rangle = \mathbf{f}^\top \mathbf{v}$, $\mathbf{f} = \mu_h f$. We remark that for V_h set up on a sequence of non-uniformly refined triangulations of Ω , the l^2 inner product $\mathbf{u}^\top \mathbf{v}$ may not provide a converging approximation of (u, v) and the distinction between the two becomes crucial for the construction of converging methods.

Finally, Korn's inequalities on $V = [H^1(\Omega)]^3$ and $Z^\perp = \{v \in V; (v, z) = 0, z \in Z\}$, $Z = \{v \in V; \epsilon(v) = 0\}$ are invoked, see [9, thm 2.1] and [9, thm 2.3]. There exist a positive constant $C = C(\Omega)$ such that

$$C\|u\|_1^2 \leq \|\epsilon(u)\|^2 + \|u\|^2 \quad u \in V. \quad (2.2)$$

and

$$C\|u\|_1^2 \leq \|\epsilon(u)\| \quad u \in Z^\perp. \quad (2.3)$$

To motivate our investigations, we present three numerical examples which discuss performance of CG and pinpointing for solving (1.1). That the pinpointing can be a suitable method for treating a singular problem is shown in the first example which considers the Poisson problem with Neumann boundary conditions. However, the method is not a cure-all as the second example shows that it does not work well for (1.1). In the third example, the singular elasticity problem is therefore solved with preconditioned CG. The employed preconditioner ignores the rigid motions leading to lack of convergence and unbounded iterations.

Bochev and Lehoucq [10] report an increase in iteration count due to pinpointing for a non-preconditioned CG in the context of singular Poisson problem. However, Krylov methods are in practice rarely applied without a preconditioner. For this reason, Example 2.1 solves the singular Poisson problem in two and three dimensions by means of pinpointing and a preconditioned CG, where algebraic multigrid (AMG) from the Hypre library [11] is used as a preconditioner. We will see that the preconditioned method yields convergent numerical solutions without increasing the iteration count.

EXAMPLE 2.1. We consider $\Omega = [0, 1]^d$, $d = 2, 3$ and the singular Poisson equation

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ \nabla u \cdot n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with unique exact solution obtained by subtracting its mean value $|\Omega|^{-1} \int_{\Omega} u \, dx$ from a manufactured u . The value of the exact solution is prescribed as a constraint for the degree of freedom at the (bottom) lower left corner of the domain, which is triangulated such that the computational mesh is refined towards the origin.

To discretize the system continuous linear Lagrange elements¹ from the FEniCS library [12] were used. The resulting linear system was solved by the preconditioned CG method implemented in the PETSc library [13], taking HypreAMG with default settings as a preconditioner. The iterations were started from a random initial guess and a relative preconditioned residual magnitude of 10^{-11} was required for convergence.

The number of iterations together with error and convergence rate based on the H^1 norm are reported in Table 2.1. Pinpointing yields numerical solutions u_h that converge with optimal rate. The number of iterations is bounded.

Table 2.1: Convergence of the pinpointing approach for the singular Poisson problem.

$d = 2$			$d = 3$		
size	$\ u - u_h\ _1$	#	size	$\ u - u_h\ _1$	#
40849	2.49E-01 (1.00)	11	12347	2.72E+00 (1.22)	10
162593	1.25E-01 (1.00)	11	92685	1.36E+00 (1.01)	11
648769	6.23E-02 (1.00)	11	718649	6.78E-01 (1.00)	12
2591873	3.11E-02 (1.00)	12	5660913	3.39E-01 (1.00)	12

Following the performance of pinpointing in the singular Poisson problem, the same approach is now applied to (1.1) in Example 2.2. Here, we will observe that fixing the solution datum in vertices of the mesh leads to slightly increased iteration counts. More importantly, we will see that the method in general does not yield converging solutions.

EXAMPLE 2.2. We consider the singular elasticity problem (1.1) with $\mu = 384$, $\lambda = 577$ and Ω obtained by rigid deformation of the box $[-\frac{1}{4}, \frac{1}{4}] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{8}, \frac{1}{8}]$. The box was first rotated around x , y and z axes by angles $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{5}$ respectively. Afterwards it was translated by the

¹Unless stated otherwise continuous linear Lagrange elements (P_1) are used to discretize all the presented numerical examples.

vector $(0.1, 0.2, 0.3)$. The unique exact solution is obtained by orthogonalizing $u = \frac{1}{4}(\sin \frac{\pi}{4}x, z^3, -y)$ with respect to the rigid motions of Ω where the orthogonality is enforced in the L^2 inner product. The solution is pictured in Figure 2.1. We note that in this example a uniform triangulation is used.

To obtain from (1.1) an invertible linear system, the exact displacement was prescribed in four different ways, cf. Table 2.2 below. $(3\circ)$ constrains six degrees of freedom in three corners of the body such that in i -th corner there are i components prescribed. This choice is motivated by the dimensionality of the space of rigid motions, cf. [3]. The fact that fixing three points in space is sufficient to prevent the body from rigid motions motivates $(1\triangleright)$ where all three components of displacement are prescribed on vertices of a single triangular element on $\partial\Omega$. However, with mesh size decreasing this constraint effectively becomes a constraint for a single (mid)point. Thus in $(3\triangleright)$ the displacement in three arbitrary triangles is fixed. Finally in $(3\bullet)$ the displacement is prescribed in three corners of the body.

The iterative solver used the same tolerances and parameters as in Example 2.1. In particular, default settings of the multigrid preconditioner were utilized and the iterations were started from random initial vectors.

The number of iterations together with error and convergence rates based on the H^1 norm are reported in Table 2.2. Note that all the considered pinpointing strategies lead to moderately increased iteration counts. The increase is most notable for $(1\triangleright)$, which effectively constrains a single point as the mesh is refined. On the other hand, strategies $(3\triangleright)$ and $(3\bullet)$, that always constrain all three components of the displacement in at least three points, yield the slowest growth rates. However, neither strategy yields convergent numerical solutions. In fact, the numerical error can often be seen to increase with resolution.

Table 2.2: Convergence of the pinpointing approach for the singular Poisson problem.

size	$3\circ$			$1\triangleright$			$3\triangleright$			$3\bullet$		
	$\ u - u_h\ _1$		#	$\ u - u_h\ _1$		#	$\ u - u_h\ _1$		#	$\ u - u_h\ _1$		#
2187	6.69E-02 (-0.02)		30	1.01E-01 (-0.70)		32	2.82E-02 (0.88)		24	2.89E-02 (0.99)		25
14739	1.27E-01 (-0.92)		35	9.61E-01 (-3.25)		40	1.08E-02 (1.38)		28	1.35E-02 (1.10)		29
107811	2.57E-01 (-1.02)		36	7.89E+00 (-3.04)		48	1.72E-02 (-0.66)		31	1.08E-02 (0.31)		32
823875	5.17E-01 (-1.01)		41	6.36E+01 (-3.01)		54	3.96E-02 (-1.21)		33	1.82E-02 (-0.75)		35

In the final example a preconditioned CG method will be applied to solve the singular elasticity problem with data such that the compatibility conditions (1.2) are met. Based on whether or not the components of the kernel are removed from the converged vector, we will see that the method yields convergent/divergent numerical solutions. We will also see that the iteration counts are not bounded.

EXAMPLE 2.3. We consider again the problem from Example 2.2. As the data satisfy (1.2), the discrete linear system is solvable and amiable to solution by the preconditioned CG method. The mass matrix is added to the singular system matrix in order to obtain a positive definite matrix in the construction of the preconditioner based on AMG. We consider two cases where the converged vector is either postprocessed by removing from it the components of the nullspace or no postprocessing is applied. We note that in this example the iterations are started from a zero initial vector and the relative tolerance of 10^{-10} is used as a convergence criterion.

The number of iterations together with error and convergence rates based on the H^1 norm are reported in Table 2.3. We observe that the method yields convergent solutions only if postprocessing is applied. This is expected as the current preconditioner introduces components of the nullspace into the solution even if the iterations are started from a right hand side and initial vector (here zero) that are orthogonal to the kernel. We note that in exact arithmetic, the CG method without

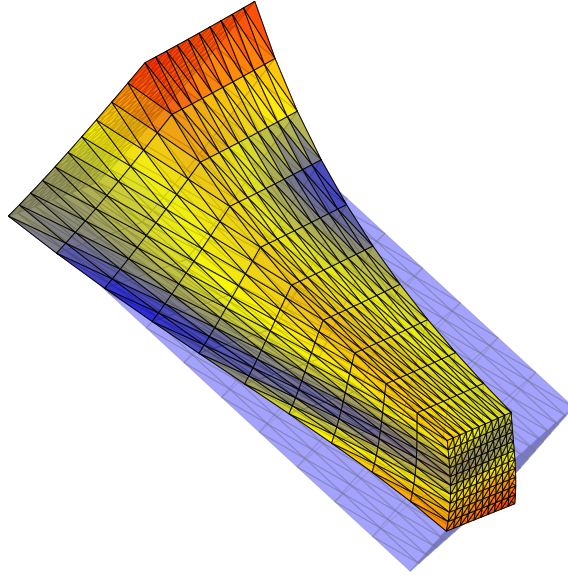


Fig. 2.1: Computational domain (blue) deformed by exaggerated(4x) analytical displacement used in the numerical examples. The deformed body is colored by the magnitude of the displacement.

preconditioner will maintain orthogonality. We further observe that the choice of preconditioner leads to unbounded iteration counts.

Table 2.3: Convergence of the preconditioned CG method with positive definite preconditioner for the singular elasticity problem.

size	kernel removed		kernel not removed	
	$\ u - u_h\ _1$	#	$\ u - u_h\ _1$	#
14739	1.14E-02 (1.09)	34	4.01E-02 (-0.12)	22
107811	5.49E-03 (1.06)	22	5.02E-02 (-0.32)	34
823875	2.71E-03 (1.02)	78	5.51E-02 (-0.13)	53
6440067	1.35E-03 (1.00)	150	5.18E-02 (0.09)	150

Examples 2.1–2.3 have illustrated some of the issues that might be encountered when solving the singular problem (1.1) with the finite element method. In particular, the following questions may be posed: (i) What is the cause of the poor convergence properties of pinpointing? (ii) What should be the optimal preconditioner for CG? (iii) What should be the optimal preconditioner for the Lagrange multiplier formulation?

With questions (ii) and (iii) answered in detail in the remainder of the text let us briefly comment on the first question. As will become apparent, the singular problem with a known kernel, such as (1.1), possesses all the information necessary to formulate a well-posed problem and a convergent numerical method. In this sense, coming up with a datum to be prescribed in the pinpointed nodes is theoretically redundant, but usually required for implementation. Further, as pointed out in [10] there are stability issues with prescribing point values of H^1 functions for $d \geq 2$. However, we note that we have not explored settings of HypreAMG that could potentially improve convergence properties of the method in Example 2.2.

3. Lagrange multiplier formulation. Let $Z \subset V = [H^1(\Omega)]^3$ denote the space of rigid motions of Ω , $m = \dim Z$. For compatible data a unique solution u of (1.1) is required to be linearly independent of functions in Z . To this end a Lagrange multiplier $p \in Q$, $Q = \mathbb{R}^m$ is introduced which enforces orthogonality of u with respect to Z . The constrained variational formulation of (1.1) seeks $u \in V, p \in Q$ such that²

$$\begin{aligned} 2\mu(\epsilon(u), \epsilon(v)) + \lambda(\nabla \cdot u, \nabla \cdot v) - p_k(v, z_k) &= (f, v) + (h, v) & v \in V, \\ -q_k(u, q_k) &= 0 & q \in Q, \end{aligned} \quad (3.1)$$

for some basis vectors $z_k \in V$, $Z = \text{span}\{z_k\}_{k=1}^m$. Equation (3.1) defines a saddle point problem for $(u, p) \in W$, $W = V \times Q$ satisfying

$$\mathcal{A} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} A & B \\ B' & \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b \\ \end{pmatrix}, \quad (3.2)$$

where $b \in V'$ such that $\langle b, v \rangle = (f, v) + (h, v)$ and operators $A : V \rightarrow V'$, $B : Q \rightarrow V'$ are defined in terms of bilinear forms

$$a(u, v) = 2\mu(\epsilon(u), \epsilon(v)) + \lambda(\nabla \cdot u, \nabla \cdot v) \quad \text{and} \quad b(u, q) = q_k(u, q_k) \quad (3.3)$$

as $\langle Au, v \rangle = a(u, v)$ and $\langle Bq, u \rangle = -b(u, q)$. We note that in (3.2) operator B' is the adjoint of B .

Existence and uniqueness of the solution to (3.2) follows from the Brezzi theory [14], see also [15, ch 3.4]. The proof shall utilize the inequalities given in Lemma 3.1.

LEMMA 3.1. *Let $u \in V$ and $\omega(u)$ be the skew symmetric part of the displacement gradient ∇u and $u_x, v_y \in V$ the rigid rotations around vectors $x, y \in \mathbb{R}^3$. Then*

$$\|\epsilon(u)\| \leq \|\nabla u\| \quad \text{and} \quad \|\omega(u)\| \leq \|\nabla u\|, \quad (3.4a)$$

$$\|\nabla \cdot u\| \leq \sqrt{3}\|\nabla u\|, \quad (3.4b)$$

$$(\omega(u), \omega(v)) = \frac{1}{2}(\nabla \times u, \nabla \times v), \quad (3.4c)$$

$$(\nabla u_x, \nabla v_y) = 2|\Omega|x^\top y. \quad (3.4d)$$

Proof. Inequalities (3.4a) follow from the orthogonal decomposition $\nabla u = \epsilon(u) + \omega(u)$. Inequalities (3.4b) and (3.4c) follow from the definitions of the terms and the Young's inequality. Finally (3.4d) is a special case of (3.4a) and (3.4c). \square

THEOREM 3.2. *Let f, h such that $b \in V'$. Then there exists a unique solution $u \in V$, $p \in Q$ of (3.2).*

Proof. We proceed by establishing the Brezzi constants. First, the bilinear form a is shown to be bounded with respect to the $\|\cdot\|_1$. Indeed, by Cauchy-Schwarz inequality and inequalities (3.4a), (3.4b)

$$\begin{aligned} a(u, v) &= 2\mu(\epsilon(u), \epsilon(v)) + \lambda(\nabla \cdot u, \nabla \cdot v) \leq 2\mu\|\epsilon(u)\|\|\epsilon(v)\| + \lambda\|\nabla \cdot u\|\|\nabla \cdot v\| \\ &\leq (2\mu + 3\lambda)\|\nabla v\|\|\nabla u\| \leq \alpha^*\|u\|_1\|v\|_1. \end{aligned}$$

with $\alpha^* = 2\lambda + 3\mu$. Ellipticity of a on $Z^\perp = \{v \in V; (v, z) = 0, z \in Z\} = \{v \in V; b(v, p), p \in Q\}$ follows from Korn's inequality (2.3). Since $\lambda \geq 0$ by assumption

$$a(u, u) = 2\mu\|\epsilon(u)\|^2 + \lambda\|\nabla \cdot u\|^2 \geq 2\mu\|\epsilon(u)\|^2 \geq \alpha_*\|u\|_1^2,$$

with $\alpha_* = 2\mu C$ and $C = C(\Omega)$ the constant from (2.3). To verify boundedness of b let $G \in \mathbb{R}^{m \times m}$ be the Gram matrix of the basis of Z with entries $G_{ij} = (z_i, z_j)$. By assumption of complete basis

²Note that (h, v) is to be understood as the L^2 inner product over $\partial\Omega$

of Z , G is a positive definite matrix. Further $G = G^\top$ and we let $0 < \lambda_* \leq \lambda^*$ be, respectively, the smallest and largest eigenvalues of G . Then

$$b(v, p) = (v, p_k z_k) \leq \|v\| \|p_k z_k\| \leq \sqrt{\lambda^*} \|v\|_1 |p|$$

and b is bounded with constant $\beta^* = \sqrt{\lambda^*}$. Finally, we show that the inf-sup property of b is satisfied. By (3.4d)

$$\sup_{v \in V} \frac{b(v, p)}{\|v\|_1} \geq \frac{(p_k z_k, p_i z_i)}{\|p_i z_i\|_1} = \frac{p^\top G p}{\sqrt{p^\top G p + 2|\Omega| p^\top D p}},$$

with $D \in \mathbb{R}^{m \times m}$ a block diagonal matrix $D = \text{diag}(I, R)$ and $R \in \mathbb{R}^{3 \times 3}$ such that $R_{ij} = e_i^\top e_j$ for axes of rigid rotations e_i . Denoting C the largest eigenvalue of the symmetric positive definite generalized eigenvalue problem for matrices D and G we have

$$\sup_{v \in V} \frac{b(v, p)}{\|v\|_1} \geq \frac{\sqrt{p^\top G p}}{\sqrt{1 + 2|\Omega|C}} \geq \sqrt{\frac{\lambda_*}{1 + 2|\Omega|C}} |p| = \beta_* |p|.$$

□

We remark that Theorem 3.2 implies that the operator $\mathcal{A} : W \rightarrow W'$ from (3.2) is an isomorphism. In particular, conditions (1.2) need not to hold for there to exist a unique solution of (3.1).

In order to find the solution of the well-posed (3.2) numerically, conditions from Theorem 3.2 must hold with discrete spaces $V_h \subset V$, $Q_h \subset Q$, see [16] or [15, ch 3.4]. Note that $Q_h = Q$ in the case studied here. Typically, satisfying the discrete inf-sup condition presents an issue and requires choice of compatible finite element discretization of the involved spaces, e.g. Taylor-Hood or MINI elements [17] for the Stokes equations. For the conforming discretization $V_h \subset V$ the following result shows that the discrete inf-sup condition holds.

THEOREM 3.3. *Let $V_h \subset V$ and b the bilinear form defined in (3.3). Then there is a constant β_* independent of h such that $\sup_{v \in V_h} \frac{b(v, p)}{\|v\|_1} \geq \beta_* |p|$.*

Proof. Since the continuous inf-sup condition holds the statement follows from Fortin's criterion [16] and we shall construct Fortin's projector $\Pi : V \rightarrow V_h$ such that $\|\Pi u\|_1 \leq C \|u\|_1$ with C independent of h and $b(u - \Pi u, q) = 0$ for any $q \in Q_h$. For given $u \in V$ we consider $u_h = \Pi u \in V_h$ the satisfies

$$2\mu(\epsilon(u_h), \epsilon(v)) + \lambda(\nabla \cdot u_h, \nabla \cdot v) + (u_h, v) = 2\mu(\epsilon(u), \epsilon(v)) + \lambda(\nabla \cdot u, \nabla \cdot v) + (u, v) \quad v \in V_h.$$

Then, testing the equation with $z_h \in V_h$, an interpolant of $z \in Z$ in V_h , gives $(u - u_h, z_h) = 0$ and in turn $b(u - \Pi u, q) = 0$. Moreover by Korn's inequality (2.2)

$$\begin{aligned} 2\mu(\epsilon(u_h), \epsilon(u_h)) + \lambda(\nabla \cdot u_h, \nabla \cdot u_h) + (u_h, u_h) &\geq 2\mu(\epsilon(u_h), \epsilon(u_h)) + (u_h, u_h) \\ &\geq C \min(2\mu, 1) \|u_h\|_1^2 = c \|u_h\|_1^2, \end{aligned}$$

while the estimate

$$2\mu(\epsilon(u), \epsilon(u_h)) + \lambda(\nabla \cdot u, \nabla \cdot u_h) + (u, u_h) \leq \max(2\mu + 3\lambda, 1) \|u\|_1 \|u_h\|_1 = C \|u\|_1 \|u_h\|_1$$

follows from (3.4a), (3.4b). Thus $\|u_h\|_1 \leq \frac{C}{c} \|u\|_1$ and the projector is bounded. □

Following Theorems 3.2, 3.3 and operator preconditioning [8, 18] the Riesz map $\mathcal{B}_1 : W' \rightarrow W$ with respect to the W inner product $(u, v)_1 + p^\top q$ with $(u, p), (v, q) \in W$

$$\mathcal{B}_1 = \begin{pmatrix} H & \\ & I \end{pmatrix}^{-1}, \quad H : V \rightarrow V', \langle H u, v \rangle = (u, v)_1 \quad \text{and} \quad I : Q \rightarrow Q, I_{ij} = \delta_{ij}, \quad (3.5)$$

defines a preconditioner for discretized (3.2) whose condition number is independent of h . This follows from Brezzi constants in Theorems 3.2, 3.3 being free of the discretization parameter. However the constants depend on the skewness of the basis of Z and material parameters. To remove the former dependency, an orthonormal basis of the space of rigid motions shall be constructed.

3.1. Construction for orthonormal basis of rigid motions. Consider a unit cube $\Omega = [-\frac{1}{2}, \frac{1}{2}]^3$ centered at the origin. Denoting $e_i, i = 1, 2, 3$ the canonical unit vectors the set

$$Z_{\square} = \{e_1, e_2, e_3, x \wedge e_1, x \wedge e_2, x \wedge e_3\}$$

constitutes an orthonormal basis of the rigid motions of Ω with respect to the L^2 inner product. Clearly, the basis for an arbitrary body can be obtained from Z_{\square} by a Gram-Schmidt process. However, we shall advocate here a construction derived from physical considerations. The construction was originally presented by the authors in [19].

LEMMA 3.4. *Let $c = |\Omega|^{-1}(x, 1)$ be the center of mass of Ω , I_Ω the tensor of inertia [20, ch 4.] of Ω with respect to c*

$$I_\Omega = \int_{\Omega} I(x - c)^\top (x - c) + (x - c) \otimes (x - c) dx$$

and $(\lambda_i, v_i), i = 1, 2, 3$ the eigenpairs of the tensor. Then the set

$$Z_\Omega = \{|\Omega|^{-\frac{1}{2}}v_1, |\Omega|^{-\frac{1}{2}}v_2, |\Omega|^{-\frac{1}{2}}v_3, \lambda_1^{-\frac{1}{2}}(x - c) \wedge v_1, \lambda_2^{-\frac{1}{2}}(x - c) \wedge v_2, \lambda_3^{-\frac{1}{2}}(x - c) \wedge v_3\} \quad (3.6)$$

is the L^2 orthonormal basis of rigid motions of Ω .

Proof. Note that by construction I_Ω is a symmetric positive definite tensor. Thus $\lambda_i > 0$ and there exists a complete set of eigenvectors $v_i^\top v_j = \delta_{ij}$. We proceed to show that the Gram matrix of the proposed basis is an identity. First $(v_i, v_j) = |\Omega|\delta_{ij}$ by orthonormality of the eigenvectors. Further, for $((x - c) \wedge v_i, v_j) = (v_i \wedge v_j, (x - c))$ and in the nontrivial case $i \neq j$ the product is zero since c is the center of mass. Finally $((x - c) \wedge v_i, (x - c) \wedge v_j) = v_i^\top I_\Omega v_j = \lambda_i \delta_{ij}$. \square We remark that the rigid motions of the body are in the constructed basis given in terms of translations along and rotations around the principal axes of the tensor that describes its rotational kinetic energy.

Note also, that the construction can be generalized to yield an orthonormal basis with respect to different inner products. In particular, let $Z_h = \text{span}\{z_k\}_{k=1}^m \subset V_h$ be functions approximating Z . For $u_h \in V_h$ let $\mathbf{u} = \pi_h u$ be a coefficient vector in the nodal basis of V_h . The l^2 orthonormal basis of Z_h can be created using Lemma 3.4 by replacing (u, v) with $\mathbf{u}^\top \mathbf{v}$. The differences between the bases are shown in Figure 3.1 where the defining principal axes of the L^2 and l^2 orthonormal basis of rigid motions are drawn. If Ω is uniformly triangulated the bases are practically identical. However, the l^2 basis changes in the presence of a non-uniform mesh refinement.

The assumption of orthonormal basis in (3.1) modifies the Brezzi constants in Theorem 3.2 (and Theorem 3.3). More specifically the Gram matrix of Z becomes identity and $\beta^* = 1$ while β_* newly depends only on the domain size. In turn, if (3.1) is considered with the orthonormal basis of the space of rigid motions, then \mathcal{B}_1 (see (3.5)) is a suitable preconditioner for (3.2) with a condition number dependent only on the geometry and material parameters.

To address the dependence on material parameters, we shall at first assume that μ and λ are comparable in magnitude. The case $\lambda \gg \mu$ is postponed until §6.

3.2. Robust preconditioning of the singular problem. Parameter robust preconditioners for the Lagrange multiplier formulation of the singular elasticity problem (3.1) can be analyzed by the operator preconditioning framework of [8]. The preconditioners are constructed by considering (3.2) in parameter dependent spaces, e.g. [21], which are equivalent with V as a set, but

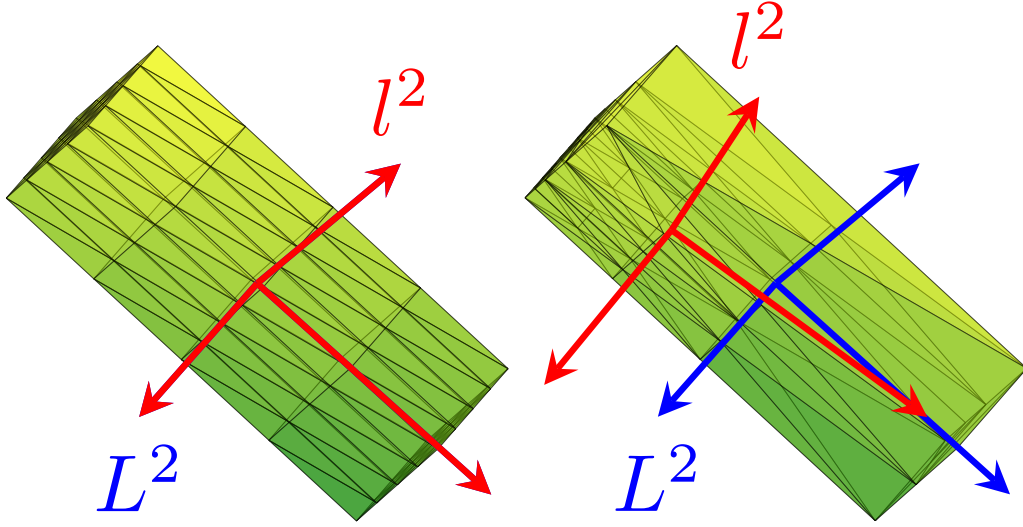


Fig. 3.1: Computational domains considered in the numerical examples for linear elasticity are obtained by uniformly refining the parent mesh. (Left) Parent is close to uniformly triangulated. (Right) The parent mesh is refined near a single edge of the domain. The blue and red arrows indicate the principal axes of the tensor I_Ω , cf. Lemma 3.4, defined using the L^2 and l^2 inner products. Axes are drawn from the center of mass computed using the respected inner products. Only the L^2 basis is stable upon change of triangulation from uniform (left) to nonuniform (right).

the topology of the spaces are given by different, parameter dependent, norms. Two such norms leading to two different preconditioners are constructed next.

Let $\{z_k\}_{k=1}^m$ be the L^2 orthonormal basis of the space of rigid motions of Ω . Bilinear forms $(\cdot, \cdot)_E$, $(\cdot, \cdot)_M$ over V are defined in terms of A from (3.2) and operators $Y : V \rightarrow V'$, $M : V \rightarrow V'$ as

$$\begin{aligned} \langle Yu, v \rangle &= (u, z_k)(v, z_k), & (u, v)_E &= \langle Au, v \rangle + \langle Yu, v \rangle, \\ \langle Mu, v \rangle &= (u, v), & (u, v)_M &= \langle Au, v \rangle + \langle Mu, v \rangle. \end{aligned} \quad (3.7)$$

The forms (3.7) define functionals $\|\cdot\|_E$ and $\|\cdot\|_M$ over V such that

$$\|u\|_E = \sqrt{(u, u)_E} \quad \text{and} \quad \|u\|_M = \sqrt{(u, u)_M}. \quad (3.8)$$

LEMMA 3.5. *Let $\|\cdot\|_E$ and $\|\cdot\|_M$ be the functionals (3.8). Then $\|\cdot\|_E$ and $\|\cdot\|_M$ define norms on V equivalent with the H^1 norm.*

Proof. From an orthogonal decomposition of $u \in V$, $u = u - (u, z_k)z_k + (u, z_k)z_k$ into $u_{Z^\perp} = u - (u, z_k)z_k \in Z^\perp$ and $u_Z = (u, z_k)z_k \in Z$ it follows that $\|u\|_M^2 = \|u\|_E^2 + \|u_{Z^\perp}\|^2$. Together with Lemma 3.1 we thus establish

$$\|u\|_E^2 \leq \|u\|_M^2 \leq (2\mu + 3\lambda + 1)\|u\|_1^2.$$

To complete the equivalence, let $C = C(\Omega)$ be the constant from Korn's inequality (2.2). Then

$$\|u\|_M^2 \geq 2\mu\|\epsilon(u)\|^2 + \|u\|^2 \geq c\|u\|_1^2,$$

with $c = C$ for $2\mu > 1$ and $c = 2\mu C$ otherwise. Finally, for equivalence of the E -norm, the Korn's inequality for $u \in Z^\perp$, see (2.3) also Theorem 3.2, yields

$$\|u\|_E^2 = 2\mu\|\epsilon(u)\|^2 + \lambda\|\nabla \cdot u\|^2 \geq 2\mu C\|u\|_1^2$$

with $C = C(\Omega)$, while from (3.4d) in Lemma 3.1

$$\|u\|_E^2 = \|u\|^2 = \frac{1}{1+2|\Omega|}\|u\|_1^2$$

for $u \in Z$. Thus E and H^1 norms are equivalent on Z^\perp and Z respectively. The proof is completed by observing that u_Z and u_{Z^\perp} are orthogonal in the E inner product

$$\begin{aligned} \|u\|_E^2 &= 2\mu\|\epsilon(u_{Z^\perp})\|^2 + \lambda\|\nabla \cdot u_{Z^\perp}\|^2 + \|u_Z\|^2 \geq 2\mu C\|u_{Z^\perp}\|_1^2 + \frac{1}{1+2|\Omega|}\|u_Z\|_1^2 \\ &\geq c(\|u_{Z^\perp}\|_1^2 + \|u_Z\|_1^2), \end{aligned}$$

$c = \min(2\mu C, (1+2|\Omega|)^{-1})$, while for the H^1 inner product $\|u\|_1^2 \leq 2(\|u_{Z^\perp}\|_1^2 + \|u_Z\|_1^2)$ holds. Thus $\|u\|_E^2 \geq \frac{c}{2}\|u\|_1^2$, for $u \in V$. \square

Using equivalent norms of V from Lemma 3.5 we readily establish equivalent norms for the product space $W = V \times Q$

$$\|w\|_E = \|(u, p)\|_E = \sqrt{\|u\|_E^2 + p^\top q} \quad \text{and} \quad \|w\|_M = \|(u, p)\|_M = \sqrt{\|u\|_M^2 + p^\top q} \quad (3.9)$$

and consider as preconditioners for (3.2) the operators $\mathcal{B}_E : W' \rightarrow W$ and $\mathcal{B}_M : W' \rightarrow W$

$$\mathcal{B}_E = \begin{pmatrix} A + Y & \\ & I \end{pmatrix}^{-1} \quad \text{and} \quad \mathcal{B}_M = \begin{pmatrix} A + M & \\ & I \end{pmatrix}^{-1}. \quad (3.10)$$

Note that the mappings (3.10) are the Riesz maps with respect to the inner products which induce norms (3.9). We proceed with analysis of the properties of \mathcal{B}_E .

THEOREM 3.6. *Let $\{z_k\}_{k=1}^m$ be the L^2 orthonormal basis of the space of rigid motions of Ω , $\mathcal{A} : W \rightarrow W'$ from (3.2) and W_E the space W considered with $\|\cdot\|_E$ norm (3.9). Then $\mathcal{A} : W_E \rightarrow W'_E$ is an isomorphism. Moreover the Riesz map $\mathcal{B}_E : W'_E \rightarrow W_E$ in (3.10) defines the canonical preconditioner for (3.2).*

Proof. We shall show that the first assertion holds by establishing the Brezzi constants. Recall the definition of the bilinear form a given in (3.3). Then, by the Cauchy-Schwarz inequality and (3.4a) in Lemma 3.1, the inequality $a(u, v) \leq \sqrt{a(u, u)}\sqrt{a(v, v)}$ holds. In turn

$$a(u, v) \leq \sqrt{a(u, u)}\sqrt{a(v, v)} \leq \sqrt{a(u, u) + (u, z_k)(u, z_k)}\sqrt{a(v, v) + (v, z_k)(v, z_k)} = \|u\|_E\|v\|_E$$

and a is bounded with respect to E norm with a constant $\alpha^* = 1$. Further $(u, z_k) = 0$ for $u \in Z^\perp$. Hence $a(u, u) = a(u, u) + (u, z_k)(u, z_k) = \|u\|_E^2$ and the form is E elliptic on Z^\perp with constant $\alpha^* = 1$. To compute the boundedness constant of the form b , the orthogonal decomposition $u = u_Z + u_{Z^\perp}$ is used together with equality $(p_i z_i, p_j z_j) = |p|^2$ which is due to orthonormality of the basis of the space of rigid motions. In turn

$$b(u, q) = q_k(u, z_k) = (u_Z, q_k z_k) \leq \|u_Z\| \|q_k z_k\| = \sqrt{a(u, u) + (u, z_k)(u, z_k)}|p| = \|u\|_E|p|.$$

We have $\beta^* = 1$. Finally, $\beta_* = 1$ in the inf-sup property

$$\sup_{u \in V} \frac{b(u, q)}{\|u\|_E} \geq \frac{(p_k z_k, p_i z_i)}{\sqrt{a(p_k z_k, p_i z_i) + (p_k z_k, p_i z_i)}} \geq \frac{|p|^2}{\sqrt{0 + |p|^2}} = |p|.$$

As all the constants are independent of material parameters, the second assertion follows from the first one by operator preconditioning [8, ch 5.]. \square Using Theorem 3.6 it is readily established that the condition number of the composed operator $\mathcal{B}_E \mathcal{A} : W \mapsto W$ is equal to one. We further note that discretizing operator \mathcal{B}_E leads to discrete nullspace preconditioners of [22, ch 6.].

While the spectral properties of \mathcal{B}_E are appealing, the preconditioner is impractical. Consider \mathbf{B}_E as a matrix representation of the Galerkin approximation of \mathcal{B}_E in $W_h \subset W$. Then $\mathbf{B}_E = \text{diag}(\mathbf{A} + \mathbf{Y}\mathbf{Y}^\top, I)^{-1}$ where $\mathbf{Y} = \mathbb{R}^{n \times m}$, $\mathbf{y}_i = \text{col}_i \mathbf{Y} = \pi_h z_k$ and $z_k \in V_h$ is the basis function of the space of rigid motions. Due to the second (nonlocal) term the matrix $\mathbf{A} + \mathbf{Y}\mathbf{Y}^\top$ is dense. Further, as shall be discussed in §4, inverting the operator requires computing (the action of) the pseudoinverse of the singular matrix \mathbf{A} . The mapping \mathcal{B}_M , on the other hand, leads to a more practical preconditioner.

THEOREM 3.7. *Let $2\mu \geq 1$, $\{z_k\}_{k=1}^m$ be the L^2 orthonormal basis of the space of rigid motions of Ω , $\mathcal{A} : W \rightarrow W'$ defined in (3.2) and W_M defined analogically to Theorem 3.6. Then $\mathcal{A} : W_M \rightarrow W'_M$ is an isomorphism. Moreover the Riesz map $\mathcal{B}_M : W'_M \rightarrow W_M$ in (3.10) defines a parameter robust preconditioner for (3.2).*

Proof. As in the proof of Theorem 3.6 we establish that $a(u, v) \leq \|u\|_M \|v\|_E$ and $b(v, p) \leq \|v\| \|p_k z_k\| \leq \|v\|_M |p|$. Setting $v = p_k z_k$ orthonormality of the basis yields $\inf_{p \in Q} \sup_{v \in V} \frac{b(v, p)}{\|v\|_M} \geq 1$. For M ellipticity of a on Z^\perp , assume existence of $C = C(\Omega)$ such that $\|u\|^2 \leq C \|\epsilon(u)\|^2$ for $u \in Z^\perp$. Then on Z^\perp

$$\|u\|^2 \leq C \|\epsilon(u)\|^2 \leq C\mu \|\epsilon(u)\|^2 \leq C(2\mu \|\epsilon(u)\|^2 + \lambda \|\nabla \cdot u\|^2) = C \|u\|_E^2$$

and

$$\|u\|_M^2 = \|u\|_E^2 + \|u\|^2 \leq (C + 1) \|u\|_E^2$$

so that $a(u, u) = \|u\|_E^2 \geq (1 + C)^{-1} \|u\|_M^2$. Finally we comment on the assumption of existence of the constant C . Assume the contrary. Then there is $u \in Z^\perp$ such that $\|e(u)\| = 1$, $\|w(u)\| = 0$ and the $\|u\|$ unbounded. However, such u violates Korn's inequality (2.3). \square

We remark that Theorem 3.7 required an additional assumption $2\mu \geq 1$. The assumption is not restrictive as it can be always achieved by scaling the equations such that the inequality is satisfied. Note also that the discrete preconditioner based on \mathcal{B}_M is such that $\mathbf{B}_M^{-1} = \text{diag}(\mathbf{A} + \mathbf{M}, I)$, with \mathbf{M} the mass matrix. The system to be assembled is therefore sparse.

Following Theorem 3.7 the condition number of the preconditioned operator $\mathcal{B}_M \mathcal{A} : W \rightarrow W$ depends solely on the constant C from Korn's inequality (2.3). An approximation for the constant is provided by the smallest positive eigenvalue λ_{\min}^+ of the problem

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{A} + \mathbf{M} & \\ & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}.$$

In Table A.1, Appendix A, the constant has been computed for two different domains; a cube from Example 2.2 and a hollow cylinder. In both cases $C \approx 1$ can be observed.

In order to demonstrate h robust properties of \mathcal{B}_M , the problem from Example 2.2 is discretized on $V_h \subset V$ and the resulting preconditioned linear system is solved by the minimal residual (MinRes) method [23]. Here the approximation of the preconditioner is provided by an algebraic multigrid (AMG), leading to the discrete operator

$$\mathbf{B}_M = \begin{pmatrix} \text{AMG}(\mathbf{A} + \mathbf{M}) & \\ & \mathbf{I} \end{pmatrix}.$$

The saddle point system was assembled and inverted using *cbc.block*, the FEniCS library for block matrices [24]. The results of the experiment are presented in Table 4.2. Clearly, the number of

iterations required for convergence is independent of the discretization. Moreover, the method yields numerical solutions which converge in the H^1 norm with the optimal rate³ on both the uniform and nonuniform meshes, cf. Figure 3.1.

A drawback of the Lagrange multiplier formulation is the cost of solving the resulting indefinite linear system. Let us denote κ the condition number of an indefinite matrix \mathbf{A} . Under simplifying assumptions on the spectrum [25, ch 3.2] gives the following bound on the relative error in residual r_n at step n

$$\frac{|r_n|}{|r_0|} \leq 2 \left(\frac{\kappa - 1}{\kappa + 1} \right)^{\lfloor n/2 \rfloor}.$$

The result should be contrasted with a similar one for the error e_n at the n -th step of CG method on positive definite matrix \mathbf{A} , e.g. [26, thm 38.5],

$$\frac{e_n^\top \mathbf{A} e_n}{e_0^\top \mathbf{A} e_0} \leq 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^n.$$

While the above estimates are known to give the worst case behaviour of the two methods, the faster rate of convergence of CG motivates investigating formulations of (1.1) to which the conjugate gradient method can be applied.

4. Conjugate gradient method for discrete singular problems. We consider a variational formulation of (1.1) for $u \in V = [H^1(\Omega)]^3$ such that

$$2\mu(\epsilon(u), \epsilon(v)) + \lambda(\nabla \cdot u, \nabla \cdot v) = (f, v) + (h, v) \quad v \in V. \quad (4.1)$$

Denoting $a : V \times V \rightarrow \mathbb{R}, l : V' \rightarrow \mathbb{R}$ the bilinear and linear forms defined by (4.1), we note that the problem is not well-posed in V . Indeed, the compatibility conditions (1.2) restrict the functionals for which the solution can be found to $l \in Z^0 = \{f \in V'; \langle f, z \rangle = 0, z \in Z\}$. Moreover, only the part of u in Z^\perp is uniquely determined by (4.1). More precisely we have the following result.

THEOREM 4.1. *Let $l \in Z^0$. Then there exists a unique solution of the problem*

$$\text{Find } u \in Z^\perp \text{ such that for any } v \in Z^\perp \text{ it holds that } a(u, v) = \langle l, v \rangle. \quad (4.2)$$

Proof. The complete proof can be found as Theorem 11.2.30 in [27]. Note that boundedness and ellipticity of a on Z^\perp with $\|\cdot\|_1$ are proven as part of Theorem 3.2. \square We remark that if (1.2) holds then $u \in Z^\perp$ solves (4.2) if and only if $(u, 0)$ solves the Lagrange multiplier problem (3.2). Further, the well-posed variational problem (4.2) is not suitable for discretization by the finite element method as the approximation leads to a dense linear system. A sparse discrete problem to which the conjugate gradient method shall be applied is therefore derived from (4.1).

Recall $m = \dim Z$, $n = \dim V_h$ and let $V_h = \text{span} \{\phi_i\}_{i=1}^n$. Discretizing the variational problem (4.1) leads to a linear system

$$\mathbf{A} \mathbf{u} = \mathbf{b}, \quad (4.3)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ such that $A_{ij} = a(\phi_j, \phi_i)$ and vector $\mathbf{b} \in \mathbb{R}^n$, $b_i = \langle l, \phi_i \rangle$. Note that we shall consider (4.3) for a general right hand side, that is, not necessarily a discretization of $l \in Z^0$. We proceed by reviewing properties of the discrete system.

Due to symmetry and ellipticity of the bilinear form a on Z^\perp there exists respectively m vectors \mathbf{z}_k and $n - m$ eigenpairs (γ_i, \mathbf{u}_i) , $\gamma_i > 0$ such that $\mathbf{A} \mathbf{z}_k = 0$, $\mathbf{z}_k^\top \mathbf{u}_i = 0$, $\mathbf{A} \mathbf{u}_i = \gamma_i \mathbf{u}_i$ and

³We recall that V_h is constructed from continuous linear Lagrange elements

Table 4.1: Preconditioned CG iterations on (4.3) obtained by discretization of (4.1) with problem parameters as in Example 2.2 and two preconditioners. Both systems are solved with relative tolerance of 10^{-10} . Uniform mesh is used.

size	$\mathbf{P}_Z \text{AMG}(\mathbf{A} + \mathbf{M})$				$\text{AMG}(\mathbf{A} \mathbf{Z})$			
	$\ u - u_h\ _1$	#	time [s]		$\ u - u_h\ _1$	#	time [s]	
14739	1.14E-02 (1.09)	22	0.491		1.14E-02 (1.09)	21	0.537	
107811	5.49E-03 (1.06)	23	10.17		5.49E-03 (1.06)	23	10.96	
823875	2.71E-03 (1.02)	24	103.5		2.71E-03 (1.02)	25	86.51	
6440067	1.35E-03 (1.00)	26	1580		1.35E-03 (1.00)	26	911.9	

$\mathbf{u}_i^\top \mathbf{u}_j = \delta_{ij}$. From the decomposition of \mathbf{A} it follows that the system (4.3) is solvable if and only if $\mathbf{z}_k^\top \mathbf{b} = 0$ for any k and the unique solution of the system is $\mathbf{u} \in \text{span}\{\mathbf{u}_i\}_{i=1}^{n-m}$. We note that the last statement is the Fredholm alternative for (4.3). As a further consequence of the decomposition it is readily verified that given compatible vector \mathbf{b} , the solution of (4.3) is $\mathbf{u} = \mathbf{B}_A \mathbf{b}$ with \mathbf{B}_A such that $\mathbf{B}_A \mathbf{y} = \sum_i \gamma_i^{-1} (\mathbf{u}_i^\top \mathbf{y}) \mathbf{u}_i$. The matrix \mathbf{B}_A is the pseudoinverse [28] or natural inverse [29, ch 3.] of \mathbf{A} .

We note that any vector from \mathbb{R}^n can be orthogonalized with respect to the kernel of \mathbf{A} by a projector $\mathbf{P}_Z = \mathbf{I} - \mathbf{Z}\mathbf{Z}^\top$, where $\mathbf{Z} \in \mathbb{R}^{n \times m}$ is the matrix consisting of l^2 orthonormal basis vectors of the kernel.

With \mathbf{b} such that $\mathbf{Z}^\top \mathbf{b} = 0$ the solution \mathbf{u} of linear system (4.3) can be computed by the conjugate gradient method, e.g. [30]. Let \mathbf{u}^0 be the starting vector for the iterations. Then, assuming exact arithmetic and no preconditioner, the method preserves the component of \mathbf{u}^0 in \mathbf{Z} , i.e. $\mathbf{Z}^\top \mathbf{u}^0 = \mathbf{Z}^\top \mathbf{u}$. In particular, $\mathbf{Z}^\top \mathbf{u}^0 = 0$ is required to obtain a solution orthogonal to the kernel. On the other hand, let \mathbf{B} be the CG preconditioner. Then the iterations introduce components of the kernel to the solution even if $\mathbf{Z}^\top \mathbf{u}^0 = 0$, unless the range of \mathbf{B} is orthogonal to \mathbf{Z} .

4.1. Preconditioned CG for singular elasticity problem. A suitable preconditioner for (4.3) is obtained by a composition with the \mathbf{P}_Z projector and we shall consider $\mathbf{B}_M = \mathbf{P}_Z(\mathbf{A} + \mathbf{M})^{-1}$ where \mathbf{M} is the mass matrix. That the preconditioner leads to bounded iteration count (and converging numerical solutions) is demonstrated in Table 4.1, cf. left pane. The preconditioner is also compared with a different preconditioner based on the approximation of the pseudoinverse \mathbf{B}_A . The approximation can be constructed by passing a kernel of the operator to the multigrid preconditioner, in the form of the l^2 orthonormal basis vectors, see [13]. Note that the preconditioners perform similarly in terms of iteration count, however, for large systems the pseudoinverse is cheaper.

We remark that in terms of operator preconditioning, the preconditioner based on the pseudoinverse can be interpreted as a Riesz map $Z^0 \rightarrow Z^\perp$ defined with respect to the inner product induced by the bilinear form a . Recall that a is symmetric and elliptic on Z^\perp . On the other hand \mathbf{B}_M approximates a mapping $Z^0 \rightarrow V \rightarrow Z^\perp$.

Having established preconditioners for the indefinite system stemming from the Lagrange multiplier formulation (3.2) and the positive semi-definite problem stemming from (4.1), we shall finally discuss approximation properties of the computed solutions. To this end the problem from Example 2.2 is considered with f perturbed by rigid motions. Note that with the new functional l the problem (3.2) is well-posed while in (4.3) a compatible right hand side \mathbf{b} will be obtained by projector \mathbf{P}_Z .

Results of the experiment are listed in Table 4.2. The Lagrange multiplier method converges with an optimal rate on both the uniformly and non-uniformly discretized mesh, cf. Figure 3.1. On the other hand, solutions to (4.3) converge to the true solution *only* on the uniform mesh while there is no convergence with nonuniform discretization. Note that this is not signaled by growth of

Table 4.2: (top) Convergence properties of the Lagrange multiplier formulation (3.2) and (bottom) the singular formulation (4.1) utilizing l^2 orthogonal basis of the nullspace to invert the system (4.3). Only the multiplier formulation yields solutions converging on uniform and nonuniform meshes. Relative tolerances of 10^{-11} and 10^{-10} are used for MinRes and CG respectively.

uniform				refined			
size	$\ u - u_h\ _1$	#	$\max_Z (u_h, z) $	size	$\ u - u_h\ _1$	#	$\max_Z (u_h, z) $
14745	1.03E-02 (1.14)	44	3.54E-07	13080	3.11E-02 (0.99)	50	1.68E-07
107817	4.84E-03 (1.09)	45	2.77E-06	98052	1.41E-02 (1.14)	53	6.73E-08
823881	2.36E-03 (1.03)	45	1.38E-06	759546	6.53E-03 (1.11)	54	8.11E-07
6440073	1.18E-03 (1.00)	44	1.75E-05	5978835	3.20E-03 (1.03)	55	2.94E-06
14739	1.14E-02 (1.09)	21	1.30E-03	13074	5.51E-02 (0.45)	26	6.06E-03
107811	5.49E-03 (1.06)	23	6.66E-04	98046	5.05E-02 (0.12)	27	6.32E-03
823875	2.71E-03 (1.02)	25	3.36E-04	759540	5.00E-02 (0.02)	29	6.43E-03
6440067	1.35E-03 (1.00)	26	1.69E-04	5978829	4.98E-02 (0.01)	31	6.49E-03

the iterations - for both methods the iteration counts are bounded. Note also that MinRes takes about twice as many iterations as CG.

From the experiment we conclude that the conjugate gradient method for (4.3), as applied so far, in general does not yield converging numerical solutions of (4.1). It is next shown that the issue is due projector $\mathbf{P}_Z = \mathbf{I} - \mathbf{Z}\mathbf{Z}^\top$ which the method uses and which is derived from the discrete problem. In particular, we show that \mathbf{P}_Z is not a correct discretization of a projector used in the continuous problem (4.2) (and (3.2)). Following the continuous problem, a modification to CG is proposed, which leads to a converging method.

4.2. Conjugate gradient method with Z^0 , Z^\perp projectors. Consider the variational problem (4.2) which was proven well-posed in Theorem 4.1 under the assumptions $l \in Z^0 \subset V'$ and $u \in Z^\perp \subset V$. In this respect, there are two subspaces associated with (4.2) and we shall define two projectors $P : V \rightarrow Z^\perp$, $P' : V' \rightarrow Z^0$ such that for $v \in V$

$$\begin{aligned} (Pu, v) &= (u, v) - (u, z_k)(v, z_k), \\ \langle P'f, v \rangle &= \langle f, v \rangle - \langle f, z_k \rangle (v, z_k), \end{aligned} \quad (4.4)$$

where $Z = \text{span}\{z_k\}_{k=1}^m$ is the L^2 orthonormal basis of the space of rigid motions (e.g. constructed by Lemma 3.4). Similar projectors were discussed in [10] for the singular Poisson problem. We note that $\langle f, Pu \rangle = \langle P'f, u \rangle$ and thus P' is the adjoint of P . Note also that the two projectors are present in the multiplier formulation (3.2).

LEMMA 4.2. *Let $f \in V'$ and P, P' be the projectors (4.4). Then $(u, p) \in V \times Q$ solves (3.2) with the right hand side $(v, q) \mapsto \langle f, v \rangle + \langle 0, q \rangle$ if and only if $u \in Z^\perp$ and u solves (4.2) with the right hand side $P'(f)$.*

Proof. It suffices to establish the relation between the right hand sides. Using orthogonality of the basis it follows from testing (3.2) with $(z_k, 0)$ that $p_k = \langle f, z_k \rangle$. Substituting the obtained Lagrange multiplier, the new right hand side of (3.2) is $(v, q) \mapsto \langle f, v \rangle - \langle f, z_k \rangle (v, z_k) + \langle 0, q \rangle = \langle P'(f), v \rangle + \langle 0, q \rangle$. \square

To derive a matrix representation of the projectors with respect to nodal basis $V_h = \text{span}\{\phi_i\}_{i=1}^n$, the mappings $\pi_h : V_h \rightarrow \mathbb{R}^n$ (the nodal interpolant) and $\mu_h : V'_h \rightarrow \mathbb{R}^n$ from (2.1) are used. We recall that $(u, v) = \mathbf{v}^\top \mathbf{M} \mathbf{u}$ for $\mathbf{u} = \pi_h u$, $\mathbf{v} = \pi_h v$ and \mathbf{M} , $M_{ij} = (\phi_j, \phi_i)$ the mass matrix while $\langle f, v \rangle = \mathbf{f}^\top \mathbf{v}$ with $\mathbf{f} = \mu_h f$. Finally, matrix $\mathbf{Y} = \mathbb{R}^{n \times m}$ is such that $\mathbf{y}_i = \text{col}_i \mathbf{Y} = \pi_h z_k$ where $z_k \in V_h$ belongs to the L^2 orthogonal basis of the space of rigid motions. Then

$$\begin{aligned} \mathbf{v}^\top \mathbf{M} \mathbf{P} \mathbf{u} &= (Pu, v) = (u, v) - (u, z_k)(v, z_k) = \mathbf{V}^\top \mathbf{M} (\mathbf{I} - \mathbf{Y} \mathbf{Y}^\top \mathbf{M}) \mathbf{u}, \\ \mathbf{f}^\top \mathbf{P}'^\top \mathbf{v} &= \langle f, Pv \rangle = \langle f, v \rangle - \langle f, z_k \rangle (v, z_k) = \mathbf{f}^\top (\mathbf{I} - \mathbf{Y} \mathbf{Y}^\top \mathbf{M}) \mathbf{v} \end{aligned} \quad (4.5)$$

Table 4.3: Convergence of conjugate gradient solutions for (4.3) with different combinations of right hand (horizontal) side and left hand side (vertical) projectors. The problem from Example 2.2 is considered. Preprocessing the right hand side and postprocessing the solution by projectors $(\mathbf{P}^\top, \mathbf{P})$ yields solutions converging with optimal rate.

	size	\mathbf{P}_Z			\mathbf{P}^\top		
		$\ u - u_h\ _1$	#	$\max_Z (u_h, z) $	$\ u - u_h\ _1$	#	$\max_Z (u_h, z) $
\mathbf{P}_Z	13074	5.51E-02 (0.45)	26	6.06E-03	5.53E-02 (0.44)	27	6.05E-03
	98046	5.05E-02 (0.12)	27	6.32E-03	5.11E-02 (0.12)	28	6.31E-03
	759540	5.00E-02 (0.02)	29	6.43E-03	5.06E-02 (0.01)	29	6.42E-03
	5978829	4.98E-02 (0.01)	31	6.49E-03	5.05E-02 (0.00)	31	6.48E-03
\mathbf{P}	13074	3.13E-02 (0.98)	27	6.84E-16	3.11E-02 (0.99)	25	6.15E-16
	98046	1.45E-02 (1.11)	28	2.94E-14	1.41E-02 (1.14)	27	2.92E-14
	759540	6.92E-03 (1.07)	29	6.39E-14	6.53E-03 (1.11)	29	6.40E-14
	5978829	3.63E-03 (0.93)	31	2.89E-13	3.20E-03 (1.03)	31	2.86E-13

and $\mathbf{P} = (\mathbf{I} - \mathbf{Y}\mathbf{Y}^\top\mathbf{M})$ is the representation of P while P' is represented by \mathbf{P}^\top . We remark that in addition to \mathbf{Y} , the rigid motions $Z_h = \text{span}\{z_k\}_{k=1}^m$ can be represented in \mathbb{R}^n by an additional matrix $\mathbf{W} = \mathbf{M}\mathbf{Y}$, which is μ_h applied to functionals $v \mapsto (z_k, v)$. Following [8] the matrices \mathbf{Y} , \mathbf{W} are termed respectively the primal and dual representation of Z_h . Observe that in (4.5) matrix \mathbf{P} uses the primal representation for \mathbf{u} while the vector is expanded in the dual representation by \mathbf{P}' . Moreover, L^2 orthogonality of Z_h yields $\mathbf{y}_i^\top \mathbf{w}_j = \delta_{ij}$. Finally note that the projectors \mathbf{P}^\top , \mathbf{P} are implicitly present in the linear system which is the discretization of the multiplier problem (3.2) with the orthogonal basis of rigid motions

$$\begin{pmatrix} \mathbf{A} & \mathbf{W} \\ \mathbf{W}^\top & \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}. \quad (4.6)$$

Indeed, $\mathbf{p} = \mathbf{Y}^\top \mathbf{b}$ from premultiplying the first equation by \mathbf{Y}^\top . Upon substitution the equation reads $\mathbf{A}\mathbf{u} = \mathbf{b} - \mathbf{W}\mathbf{Y}^\top \mathbf{b} = \mathbf{P}^\top \mathbf{b}$. Further the solution is such that $\mathbf{P}\mathbf{u} = \mathbf{0}$.

The situation where the continuous problems (3.2), (4.2) and the discrete problem (4.6) use different projectors for the left and right hand sides contrasts with (4.3) which utilizes \mathbf{P}_Z to obtain consistent right hand side and the solution is such that $\mathbf{P}_Z \mathbf{u} = \mathbf{0}$ as well. This observation together with the lack of convergence of the CG method, cf. Table 4.2, motivate that the CG method on (4.3) is used with the following two modifications: (i) the iterations are started from vector $\mathbf{P}^\top \mathbf{b}$, (ii) \mathbf{P} is applied to the final solution.

The effect of the proposed modifications is shown in Table 4.3. The problem from Example 2.2 is considered on a non-uniform mesh and CG on (4.3) is applied with different combinations of projectors used to obtain the right hand side from incompatible vector \mathbf{b} and to orthogonalize the converged solution. We observe that only the case $(\mathbf{P}^\top, \mathbf{P})^4$ yields optimal convergence. With $(\mathbf{P}_Z, \mathbf{P})$ the rate is slightly smaller than one. In the remaining two cases the solution do not converge suggesting that for convergence \mathbf{P} must be applied to the solution.

The results shown in Table 4.3 are satisfactory in a sense that preprocessing the right hand side with \mathbf{P}^\top and postprocessing the solution with \mathbf{P} improved the convergence properties of the CG method for (4.3). However, the modifications alter the original discrete problem and thus the properties of the new problem should be discussed. We note that in the discussion \mathbf{Z} , \mathbf{Y} are respectively \mathbf{I} and \mathbf{M} orthogonal basis of the nullspace of \mathbf{A} . Further, the transformation matrix between the basis is $c \in \mathbb{R}^{m \times m}$ such that $\mathbf{Z} = \mathbf{Y}c$ and we have $\mathbf{Y}^\top \mathbf{M} \mathbf{Z} = c$.

⁴ Elements of the tuple denote respectively the projector for the right hand side and the left hand side.

First, admissibility of the modified right hand side $\mathbf{P}^\top \mathbf{b}$ is considered. Using the transformation matrix it holds that $\mathbf{Z}^\top \mathbf{P}^\top \mathbf{b} = 0$ and thus $\mathbf{P}^\top \mathbf{b}$ is compatible and the solution can be obtained by a pseudoinverse (or equivalently by CG). The computed solution of the new linear system then satisfies $\mathbf{Z}^\top \mathbf{u} = 0$. However, the continuous problem requires orthogonality $\mathbf{Y}^\top \mathbf{M} \mathbf{u} = Ch$. As the two conditions are related through $|\mathbf{Y}^\top \mathbf{M} \mathbf{u}|^2 = \mathbf{u}^\top \mathbf{M} \mathbf{Z} (c^\top c)^{-1} \mathbf{Z}^\top \mathbf{M} \mathbf{u} = \mathbf{u}^\top \mathbf{M} \mathbf{Z} (\mathbf{Z}^\top \mathbf{M} \mathbf{Z})^{-1} \mathbf{Z}^\top \mathbf{M} \mathbf{u}$, and $\mathbf{Z}^\top \mathbf{Z} = \mathbf{I}$, orthogonality in the L^2 inner product depends on similarity of the mass matrix with identity. This is essentially a condition on the mesh and $|\mathbf{Y}^\top \mathbf{M} \mathbf{Z}| \geq C$ is possible (as observed in Table 4.3).

To enforce orthogonality constraint $\mathbf{Y}^\top \mathbf{M} \mathbf{u} = 0$ without postprocessing we shall finally consider linear system $\mathbf{A} \mathbf{u} = \mathbf{P}^\top \mathbf{b}$ and require $\mathbf{P} \mathbf{u} = 0$ for uniqueness. In this case the solution is not provided by pseudoinverse \mathbf{B}_A . However, a similar construction based on the generalized eigenvalue problem can be used instead.

LEMMA 4.3. *Let \mathbf{u} be a unique solution of $\mathbf{A} \mathbf{u} = \mathbf{P}^\top \mathbf{b}$, satisfying $\mathbf{P} \mathbf{u} = 0$ and $\mathbf{\Gamma} \in \mathbb{R}^{n \times n}$, $\mathbf{U} \in \mathbb{R}^{n \times n-m}$ such that $\mathbf{A} \mathbf{U} = \mathbf{M} \mathbf{U}^\top$, $\mathbf{U}^\top \mathbf{M} \mathbf{U} = \mathbf{I}$. Then $\mathbf{u} = \mathbf{B} \mathbf{P}^\top \mathbf{b}$ where $\mathbf{B} = \mathbf{U} \mathbf{\Gamma}^{-1} \mathbf{U}^\top$.*

Proof. First, note that the existence of matrices \mathbf{U} , $\mathbf{\Gamma}$ follows from positive semi-definiteness of \mathbf{A} . Further, by \mathbf{M} orthogonality of the eigenvectors $\mathbf{M} \mathbf{U} \mathbf{x} = \mathbf{P}^\top \mathbf{b}$ holds with $\mathbf{x} = \mathbf{U}^\top \mathbf{b}$. As $\mathbf{Y}^\top \mathbf{M} \mathbf{U} = 0$ any vector $\mathbf{B} \mathbf{b}$ is \mathbf{M} orthogonal with \mathbf{Y} and thus $\mathbf{P} \mathbf{B} \mathbf{b} = 0$. It remains to show that the composition $\mathbf{A} \mathbf{B}$ is the identity on the subspace spanned by columns of $\mathbf{M} \mathbf{U}$

$$\mathbf{A} \mathbf{B} \mathbf{M} \mathbf{U} = \mathbf{A} \mathbf{U} \mathbf{\Gamma}^{-1} \mathbf{U}^\top \mathbf{M} \mathbf{U} = \mathbf{A} \mathbf{U} \mathbf{\Gamma}^{-1} = \mathbf{M} \mathbf{U} \mathbf{\Gamma}^{-1} = \mathbf{M} \mathbf{U}.$$

□

5. Natural norm formulation. An attractive feature of the variational problem (4.1) is the fact that the resulting linear system is amiable to solution by the CG method, which when modified following §4 yields converging solutions. However, the projectors P' , P are only applied as pre and postprocessor and the CG loop (Lanczos process) is in this respect detached from the continuous problem. Moreover the method requires a special preconditioner that handles the nullspace of matrix \mathbf{A} . A formulation which leads to a positive definite linear system requiring only a regular (not nullspace aware) preconditioner shall be studied next.

THEOREM 5.1. *Let $a : V \times V \rightarrow \mathbb{R}$, $a(u, v) = 2\mu(\epsilon(u), \epsilon(v)) + \lambda(\nabla \cdot u, \nabla \cdot v)$ and $Z = \text{span}\{z_k\}_{k=1}^m$ the L^2 orthogonal basis of the space of rigid motions. Further let $l \in Z^\perp$. There exists a unique $u \in V$ such*

$$a(u, v) + (u, z_k)(v, z_k) = \langle l, v \rangle \quad v \in V. \quad (5.1)$$

Moreover $u \in Z^\perp$.

Proof. Recall that the bilinear form above is the inner product $(u, v)_E$ from (3.7) which induces an equivalent norm on V , cf. Lemma 3.5. The existence and uniqueness of the solution now follow from the Lax-Milgram lemma. Testing the equation with $v = z_i$ yields $(u, z_i) = 0$ and in turn $u \in Z^\perp$. □ We remark that the solution of (5.1) and (4.2) are equivalent because $l \in Z^\perp$. Note also that Theorem 3.7 gives equivalence bounds $(1 + C)^{-1} \|u\|_M^2 \leq \|u\|_E^2 \leq \|u\|_M^2$, $C = C(\Omega)$ and in turn the Riesz map with respect to the inner product $(u, v)_M = a(u, v) + (u, v)$ defines a suitable h robust preconditioner for (5.1). Finally, observe that the L^2 orthogonality of decomposition $u = u_Z + u_{Z^\perp}$, $u_Z = (u, z_k) z_k$ is respected by the inner product $(\cdot, \cdot)_E$, see (3.7). The norm $\|u\|_E$, see (3.8), thus considers Z and Z^\perp with L^2 and a induced norms which are the natural norms for the subspaces.

Using (4.5) the natural norm formulation (5.1) leads to a positive definite linear system

$$[\mathbf{A} + \mathbf{M} \mathbf{Y} (\mathbf{M} \mathbf{Y})^\top] \mathbf{u} = \mathbf{P}^\top \mathbf{b}.$$

where we recognize a dense matrix from the discretization of \mathcal{B}_E preconditioner of the Lagrange multiplier formulation, cf. Theorem 3.6. Therein the inverse of the matrix was of interest. However,

Table 5.1: Convergence study of the natural norm formulation (5.1) for the singular elasticity problem from Example 2.2. The system is solved with relative tolerance 10^{-11} . The CG iterations use a preconditioner $\text{AMG}(\mathbf{A} + \mathbf{M})$. The iteration count remains bounded and the solutions converge with the optimal rate.

uniform				refined			
size	$\ u - u_h\ _1$	#	$\max_Z (u_h, z) $	size	$\ u - u_h\ _1$	#	$\max_Z (u_h, z) $
14739	1.03E-02 (1.14)	33	2.57E-08	13074	3.11E-02 (0.99)	39	3.70E-08
107811	4.84E-03 (1.09)	29	1.80E-05	98046	1.41E-02 (1.14)	41	3.46E-08
823875	2.36E-03 (1.03)	37	9.23E-09	759540	6.53E-03 (1.11)	43	8.90E-08
6440067	1.18E-03 (1.00)	33	2.38E-05	5978829	3.20E-03 (1.03)	46	3.53E-08

relevant for the CG method here is only the matrix vector product, which can be computed efficiently by storing separately \mathbf{A} and $\mathbf{M}\mathbf{Y}$, the dual representation of rigid motions in V_h .

With (5.1) we finally revisit the test problem from Example 2.2. Results of the method are summarized in Table 5.1. Optimal convergence rate is observed with both uniform and nonuniform meshes. Moreover, the CG iteration count with the proposed Riesz map preconditioner approximated by $\text{AMG}(\mathbf{A} + \mathbf{M})$ remains bounded. An interesting observation is the fact that the error in the orthogonality constraint is smaller in comparison to the Lagrange multiplier formulation, cf. Table 4.2.

6. Nearly incompressible materials. So far we have assumed that μ and λ are comparable in magnitude. In this section we handle the case where $\lambda \gg \mu$ and the material is nearly incompressible. The variational problems (3.1), (4.1), (5.1) studied thus far were based on the pure displacement formulation of linear elasticity (1.1) and H^1 conforming finite element spaces were used for their discretization. Due to the *locking* phenomenon the approximation properties of their respected solutions are known to degrade for nearly incompressible materials with $\lambda \gg \mu$, (equivalently Poisson ratio close to $1/2$), see e.g. [15, ch 6.3]. Moreover, the incompressible limit presents a difficulty for convergence of iterative methods in the standard form.

Methods robust with respect to increasing λ can be formulated using a discretization with nonconforming elements, [27, ch 11.4]. However, this method fails to satisfy the Korn's inequality. To the authors' knowledge the only finite element method that is both robust in λ and satisfies Korn's inequality is [31, 32]. In addition to problems with the discretization, standard multigrid algorithms do not work well for large λ and special purpose algorithms must be used [33]. For this reason we resort to a more straightforward solution of the mixed formulation where an additional variable, the *solid pressure* p , is introduced. Let the solid pressure be defined as $p = \lambda \nabla \cdot u$ so that (3.1) is reformulated as

$$\begin{aligned}
\nabla \cdot (2\mu \epsilon(u)) - \nabla p &= f && \text{in } \Omega, \\
\lambda \nabla \cdot u - p &= 0 && \text{in } \Omega, \\
\sigma(u) \cdot n &= h && \text{on } \partial\Omega.
\end{aligned} \tag{6.1}$$

Note that the problem is singular, since any pair $u \in Z$, $p = 0$ can be added to the solution. In fact such pairs constitute the kernel of (6.1). To obtain a unique solution we shall as in §3, require that u is orthogonal to the rigid motions Z . We assume that the basis of Z is orthonormal.

Setting $Q = L^2(\Omega)$, $Y = \mathbb{R}^6$ we shall consider a variational problem for triplet $u \in V$, $p \in Q$, $x \in Y$ such that

$$\begin{aligned}
2\mu(\epsilon(u), \epsilon(v)) + (p, \nabla \cdot v) + x_k(v, z_k) &= (f, v) + (h, v) && v \in V, \\
(q, \nabla \cdot u) - \lambda^{-1}(p, q) &= 0 && q \in Q, \\
y_k(u, z_k) &= 0 && y \in Y.
\end{aligned} \tag{6.2}$$

Equation (6.2) is a double saddle point problem

$$\mathcal{A} \begin{pmatrix} u \\ p \\ x \end{pmatrix} = \begin{pmatrix} A & B & D \\ B' & -\lambda^{-1}C & \\ D' & & \end{pmatrix} \begin{pmatrix} u \\ p \\ x \end{pmatrix} = \begin{pmatrix} b \\ \\ \end{pmatrix},$$

with operators $A : V \rightarrow V'$, $B : Q \rightarrow V'$, $C : Q \rightarrow Q'$ and $D : X \rightarrow V'$ defined as

$$\begin{aligned} \langle Au, v \rangle &= 2\mu(\epsilon(u), \epsilon(v)), & \langle Bp, v \rangle &= (p, \nabla \cdot v), \\ \langle Cp, q \rangle &= (p, q), & \langle Dx, v \rangle &= x_k(v, z_k). \end{aligned}$$

To show well-posedness of the constrained mixed formulation (6.2) the abstract theory for saddle points problems with small (note that that $\lambda \gg 1$) penalty terms [15, ch 3.4] is applied. To this end we introduce the bilinear forms $a(u, v) = \langle Au, v \rangle$,

$$b(v, (p, x)) = \langle Bp, v \rangle + \langle Dx, v \rangle, \quad (6.3)$$

$c((p, y), (q, x)) = \langle Cp, q \rangle$ so that (6.2) is recast as

$$\begin{aligned} a(u, v) + b(v, (p, x)) &= (f, v) + (h, v) & v &\in V, \\ b(u, (q, y)) - \lambda^{-1}(p, q) &= 0 & (q, y) &\in Q \times Y. \end{aligned} \quad (6.4)$$

The space $Q \times Y$ will be considered with the norm $\|(p, x)\| = \sqrt{\|p\|^2 + |x|^2}$, while V is considered with the H^1 norm. Following [15, thm 4.11] the problem (6.4) is well-posed provided that the assumptions of Brezzi theory hold and in addition c is continuous and c and a are positive

$$a(u, u) \geq 0, \quad u \in V \quad \text{and} \quad c((p, x), (p, x)) \geq 0, \quad (p, x) \in Q \times Y.$$

We review that continuity and V -ellipticity of a on Z^\perp was shown in Theorem 3.2 and as $a(z, z) = 0$, $z \in Z$, the form is positive on V . Moreover, by Lemma 3.1, Cauchy-Schwarz inequality and orthonormality of basis

$$\begin{aligned} b(v, (p, x)) &= (p, \nabla \cdot v) + x_k(v, z_k) \leq \sqrt{3}\|p\|\|\nabla v\| + \|v\||x| \leq \sqrt{3}\sqrt{\|v\|^2 + \|\nabla v\|^2}\sqrt{\|p\|^2 + |x|^2} \\ &\leq \beta^*\|v\|_1\|(p, x)\|. \end{aligned}$$

It is easy to observe that continuity and positivity of the bilinear form c hold and thus (6.4) is well-posed provided that the inf-sup condition is satisfied. We note that the proof requires extra regularity of the boundary.

LEMMA 6.1. *Let Ω with a smooth boundary and b be the bilinear form over $V \times (Q \times Y)$ defined in (6.3). There exists $\beta_* = \beta_*(\Omega)$ such that*

$$\sup_{v \in V} \frac{b(v, (p, x))}{\|v\|_1} \geq \beta_*\|(p, x)\|.$$

Proof. Let $p \in Q$ and $x \in Y$ given. Following [27, thm 11.2.3] there exists for every p , $v^* \in V$ such that

$$p = \nabla \cdot v^*, \quad (6.5a)$$

$$\|v^*\|_1 \leq C(\Omega)\|p\|. \quad (6.5b)$$

The element v^* is constructed from the unique solution of the Poisson problem

$$\begin{aligned} -\Delta w &= p & \text{in } \Omega, \\ w &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (6.6)$$

taking $v^* = -\nabla w$. Observe that the computed $v^* \in Z^\perp$

$$-(z, v^*) = \int_{\Omega} z \nabla w = \int_{\partial\Omega} w z \cdot n - \int_{\Omega} w \nabla \cdot z = 0 \quad z \in Z. \quad (6.7)$$

Orthogonality of v^* and (6.5a) yields that $b(v^* + x_k z_k, (p, x)) = (p, \nabla \cdot v^*) + (x_k z_k, x_l z_l) = \|p\|^2 + |x|^2$. Further, by Cauchy-Schwarz and Young's inequalities

$$\begin{aligned} \|v^* + x_k z_k\|_1^2 &= \|v^* + x_k z_k\|^2 + \|\nabla(v^* + x_k z_k)\|^2 \\ &= \|v^*\|^2 + \|x_k z_k\|^2 + \|\nabla v^*\|^2 + 2(\nabla v^*, \nabla x_k z_k) + \|\nabla x_k z_k\|^2 \\ &\leq 2\|v^*\|_1^2 + 2(\|v^*\|^2 + \|\nabla x_k z_k\|^2). \end{aligned}$$

Using (6.5b) and Lemma 3.1 gives $\|v^* + x_k z_k\|_1^2 \leq 2C(\Omega)\|p\|^2 + (1 + 2|\Omega|)|x|^2 \leq c(\Omega)\|(p, x)\|^2$. Combining the observations

$$\sup_{v \in V} \frac{b(v, (p, x))}{\|v\|_1} \geq \frac{b(v^* + x_k z_k, (p, x))}{\|v^* + x_k z_k\|_1} = \frac{\|p\|^2 + |x|^2}{\|v^* + x_k z_k\|_1} \geq \frac{1}{c} \sqrt{\|p\|^2 + |x|^2} = \frac{1}{c} \|(p, x)\|.$$

□ We remark that none of the constants of the problem (6.4) depends on λ despite the norm of $Q \times Y$ being free of the parameter, cf. also [34, 35]. Observe also that with H^1 norm on V the boundedness constant of a depends on μ , cf. Theorem 3.2, and thus the parameter shall be included in the norm to get a μ independent preconditioner. Finally, note that tighter bounds (e.g. in the proof of Lemma 6.1) can be obtained if the space V is considered with the norm $u \mapsto \sqrt{\mu\|\epsilon(u)\|^2 + \|u\|^2}$.

Motivated by the above, we shall consider as the preconditioner for the well-posed problem (6.4) a Riesz map $\mathcal{B} : (V \times Q \times Y)' \rightarrow (V \times Q \times Y)$ with respect to the inner product inducing the norm $(u, p, x) \mapsto \sqrt{\mu\|\epsilon(u)\|^2 + \|u\|^2 + \|p\|^2 + |x|^2}$

$$\mathcal{B} = \begin{pmatrix} A + M & & \\ & C & \\ & & I \end{pmatrix}^{-1}, \quad (6.8)$$

where M was defined in (3.7). Similar preconditioners for the Dirichlet problem has been discussed in [35].

REMARK 6.1 (Lemma 6.1 in the discrete case). *The continuous inf-sup condition can be extended to Taylor-Hood discretizations in the following way. We consider $V_h \subset V$, $Q_h \subset Q$ approximated with the lowest order Taylor-Hood element. Given $p_h \in Q_h$ both the element $v_h^* \in V_h$ and $w_h \in Q_h$ from Lemma 6.1 are found as the solution to the mixed Poisson problem*

$$\begin{aligned} (v_h^*, v) + (\nabla_h w_h, v) &= 0 & v \in V_h, \\ (\nabla_h q, v_h^*) &= -(p_h, q) & q \in Q_h. \end{aligned}$$

The problem is well-posed due to the weak inf-sup condition

$$\sup_{v \in V_h} \frac{(v_h, \nabla_h q_h)}{\|v_h\|} \geq C\|q_h\|_1.$$

Since $z \in V_h$ a direct calculation shows that the orthogonality condition (6.7) is satisfied.

Both in the above and in the construction of the proof of Lemma 6.1 we relied on a well-posed mixed Poisson problem to obtain orthogonality with respect to the kernel. We note that stable Stokes element $P_2 - P_0$ does not allow for such a construction and does not give h uniform bounds.

To show that the preconditioner (6.8) is robust with respect to λ , the problem from Example 2.2 is considered with $\mu = 1$ and $\lambda \in [1, 10^8]$. The spaces V and Q are approximated by lowest order Taylor-Hood elements for which the discrete inf-sup condition from Lemma 6.1 holds following Remark 6.1. The non-trivial blocks of the preconditioner are inverted using algebraic multigrid and the system is solved using the MinRes method requiring reduction of the preconditioned residual norm by a factor of 10^{10} for convergence.

From the results of the experiment, summarized in Table 6.1, it is evident that the iteration count is bounded in λ as well as in the discretization parameter.

Table 6.1: Iteration counts of the preconditioned MinRes method for mixed linear elasticity problem (6.2) and different values of Lamé constant λ . The iteration counts remain bounded for the considered values of the parameter.

dim(V)	dim(Q)	λ				
		10^0	10^2	10^4	10^6	10^8
14739	729	109	113	100	70	36
107811	4913	107	109	103	69	36
823875	35937	109	109	107	72	36
6440067	274625	109	108	113	75	37

7. Conclusions. We have studied the singular Neumann problem of linear elasticity. Four different formulations of the problem have been analyzed and mesh independent preconditioners established for the resulting linear systems within the framework of operator preconditioning. We have proposed a preconditioner for the (singular) mixed formulation of linear elasticity, that is robust with respect to the material parameters. Using an orthonormal basis of the space of rigid motions, discrete projection operators have been derived and employed in a modification to the conjugate gradients method to ensure optimal error convergence of the solution.

Appendix A. Eigenvalue bounds for Lagrange multiplier preconditioners. Bounds for the eigenvalues of operators $\mathcal{B}_E \mathcal{A}$ and $\mathcal{B}_M \mathcal{A}$ from (3.2) and (3.10) are approximated by considering the eigenvalue problems

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \lambda \mathbf{B}_i^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} \quad (\text{A.1})$$

with the left hand side the discretization of (3.2) and \mathbf{B}_i , $i \in \{E, M\}$ discretizations of preconditioners \mathcal{B}_i from (3.10). The spectrum of the symmetric, indefinite problem (A.1) is a union of negative and positive intervals $[\lambda_{\min}^-, \lambda_{\max}^-]$, $[\lambda_{\min}^+, \lambda_{\max}^+]$. Following the analysis in Theorems 3.6 and 3.7 negative bounds equal to -1 are expected for both preconditioners. Further, the positive eigenvalues are bounded from above by 1. Finally, $\lambda_{\min}^+ = -1$ for \mathcal{B}_E while the constant $C = C(\Omega)$ from the Korn's inequality determines the bound for \mathcal{B}_M .

In the experiment, Ω as a cube from Example 2.2 and a hollow cylinder with inner and outer radii $\frac{1}{2}$, 1 and height 2 are considered. Lamé constants $\mu = 384$, $\lambda = 577$ are used. For both bodies $C \approx 1$ is observed, cf. Table A.1. The remaining bounds agree well with the analysis.

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Table A.1: Spectral bounds for eigenvalue problems (A.1). (Top) The body is cube. (Bottom) The body is a cylinder.

	size	κ	$\lambda_{\min}^- + 1$	$\lambda_{\max}^- + 1$	$\lambda_{\min}^+ - 1$	$\lambda_{\max}^+ - 1$
B_E	87	1.0000	-6.83E-11	2.92E-11	-4.36E-11	5.89E-12
	381	1.0000	-1.38E-10	7.00E-12	-1.61E-10	5.55E-15
	2193	1.0000	-5.88E-10	1.65E-11	-6.23E-10	9.55E-15
	14745	1.0000	-1.10E-08	-4.27E-09	-2.00E-08	1.73E-14
B_M	87	1.0001	-6.64E-11	4.46E-12	-1.10E-04	1.03E-11
	381	1.0002	-1.35E-10	-1.06E-11	-2.33E-04	-5.33E-12
	2193	1.0004	-5.73E-10	-1.12E-11	-4.00E-04	5.91E-12
	14745	1.0005	-2.37E-09	-7.73E-11	-4.97E-04	-4.47E-11
B_E	210	1.0000	-3.91E-12	-4.46E-13	-4.58E-12	9.33E-15
	462	1.0000	-3.82E-12	-8.91E-13	-4.55E-12	5.77E-15
	1764	1.0000	-9.32E-12	-4.40E-12	-1.08E-11	1.31E-14
	8292	1.0000	-3.71E-11	-1.74E-11	-4.06E-11	6.26E-14
B_M	210	1.0752	1.84E-02	7.00E-02	-7.00E-02	-2.57E-06
	462	1.0219	1.94E-03	2.14E-02	-2.14E-02	-2.21E-06
	1764	1.0069	1.14E-03	6.82E-03	-6.82E-03	-4.57E-07
	8292	1.0022	1.60E-04	1.66E-03	-2.17E-03	-2.10E-08

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